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# Squeezed spin states and Heisenberg interaction

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**Abstract.** Squeezed spin states are defined through canonical transformations analogous to the squeezed states for boson systems and variances of spin components are obtained. The defined concept of squeezed spin states has been applied to Heisenberg interaction for the spin system with $S = 1$.

## 1. Introduction

Squeezed states, which play an important role in quantum optics [1] and in condensed matter physics [2] are now also considered for spin systems [3, 4]. These states are nonclassical in the sense that uncertainty in one variance is compressed at the expense of the complementary variance of two noncommuting operators, while keeping their product at a minimum value as predicted for coherent states [5]. In the case of spin components, the $SU(1, 1)$ and $SU(2)$ coherent states were first introduced by Barut and Girardello [6] and Radcliffe [7], respectively, and coherent states have been generalized for an arbitrary Lie group by Perelomov [8]. In this study, we consider only the squeezed states related to the $SU(2)$ coherent states. The spin or angular momentum system $\vec{S} = (S_x, S_y, S_z)$ obeys the well known commutation relation $[S_i, S_j] = \mathrm{i} \in_{ijk} S_k$ where $i, j, k$ refer to the components of any orthogonal basis. The associated uncertainty relation is

$$\Delta S_i \Delta S_j \geqslant \tfrac{1}{2}|\langle S_k\rangle| \tag{1}$$

where the right-hand side is state dependent. The equality in (1) is satisfied when both sides have local minima and these states are called minimum-uncertainty states (MUS). All other states leading to an equality but without any local minimum are called intelligent states [9]. In a recent paper by Trifonov [10], a more restrictive definition has been given, in which variances of two components normal to the mean spin direction are equal. This is achieved when the state is an eigenstate of one of the operators, $S_\pm = S_x \pm \mathrm{i}S_y$. If one of the variances is now reduced below its value in the MUS at the expense of the other, one obtains squeezed spin states (SSS). According to the definition of Wodkiewicz and Eberly [3] this can be expressed as $(\Delta S_i)^2 \leqslant \frac{1}{2}|\langle S_z\rangle|$ for $i = x$ or $y$. In the case of the Trifonov definition, the variances given by $(\Delta S_x)^2 = (\Delta S_y)^2 = \frac{S}{2}$ in the MUS now become $(\Delta S_i)^2 \leqslant \frac{S}{2}$ for $i = x$ or $y$ in the SSS, where S is the total spin of the system: here the two states $|S, \pm S\rangle$ are the only spin MUS with equal variances. While these are widely accepted as two different definitions for spin squeezing, there are alternative ways to construct such states. In one approach spins can be squeezed by rotating the coherent spin states (CSS) [4]; in the other it can be accomplished by using $S_\pm$ to form the operator $J(u, v) = uS_- + vS_+$ such that $J(u, v)|u, v, s\rangle = 0$ [10].

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In this work we attempt to construct the SSS starting from the $SU(2)$ coherent states and by using a canonical transformation, which in fact lies between these two approaches. In section 2 we introduce the canonical transformation to form the SSS and calculate the uncertainties of spin components. As a nontrivial application of this model, in section 3 we consider a simple spin system with two elementary $\frac{1}{2}$ spin interacting via the anisotropic Heisenberg Hamiltonian, whose exact solution is known.

## 2. Squeezed spin states

The CSS are defined through the relation involving the lowering operator $S_-$ for spins

$$|\mu\rangle = (1+|\mu|^2)^{-S} e^{\mu S_-}|S,S\rangle \tag{2}$$

where $|S,S\rangle$ is a normalized fiducial state, and for convenience it can be chosen to be an eigenvector of $S_z$. It is also convenient to parametrize the coherent states by $\mu = \tan\frac{\theta}{2}e^{i\phi}$, where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

The $S$ spin coherent states $|\theta,\phi\rangle$ is equivalent to a set of 2S elementary $\frac{1}{2}$ spins all pointing in the same direction $(\theta,\phi)$ [4]. In our calculations we only consider two elementary spin system, i.e. $S = 1$. Thus (2) can be written as

$$|\theta,\phi\rangle = \left[|11\rangle + e^{i\phi}\tan\frac{\theta}{2}\sqrt{2}|10\rangle + e^{2i\phi}\left(\tan\frac{\theta}{2}\right)^2|1-1\rangle\right] \Big/ \left(1+\tan^2\frac{\theta}{2}\right) \tag{3}$$

where $\theta$ and $\phi$ can either take special values to form the initial state $|\frac{\pi}{2},0\rangle$ as in [4] or can be chosen as parameters to establish stable states for a pair in two- and three-dimensional lattices as in [11]. We take them as variational parameters in the energy calculation of the next section.

In order to define SSS we introduce a unitary operator

$$U = e^T \qquad \text{with} \quad T = \alpha S_+^2 - \alpha^* S_-^2 \tag{4}$$

and transform $S_i$ component of the spin in a commutator expansion of the form

$$U^{-1}S_iU = S_i + \tfrac{1}{1!}[T,S_i] + \tfrac{1}{2!}[T,[T,S_i]] + \cdots. \tag{5}$$

Here each term in the expansion is proportional to powers of $\alpha$ and $\alpha^*$. Contrary to boson squeezing, this expansion cannot be summed in a closed form due to the commutation relations between $S_i$ and $S_j$. However, it is sufficient to take a few terms in the expansion, since $|\alpha| < 1$, as will be seen below. We therefore consider three terms up to the second power of $\alpha$ and $\alpha^*$ in (5) for the calculation of variances of $S_x$, $S_y$, $S_z$ as an approximation. In this approach, the expectation values of $\tilde{S}_i$ spin components after squeezed states transformation can be easily calculated by making use of equation (3):

$$\begin{aligned}
\langle\theta,\phi|\tilde{S}_x|\theta,\phi\rangle &= \cos\phi\sin\theta - r\sin 2\theta\cos(\beta+\phi) - 2r^2\cos\phi\sin\theta \\
\langle\theta,\phi|\tilde{S}_y|\theta,\phi\rangle &= \sin\phi\sin\theta + r\sin 2\theta\sin(\beta+\phi) + 2r^2\sin\phi\sin\theta \\
\langle\theta,\phi|\tilde{S}_z|\theta,\phi\rangle &= \cos\theta + 2r\sin^2\theta\cos(\beta+2\phi) - 8r^2\cos\theta
\end{aligned} \tag{6}$$

where $\alpha = re^{i\beta}$ is used. In the same coherent states the expectation values of $\tilde{S}_i^2$ are

$$\begin{aligned}
\langle\theta,\phi|\tilde{S}_{x(y)}^2|\theta,\phi\rangle &= \sin^2\theta\{+(-)\tfrac{1}{4}\cos 2\phi + \tfrac{1}{2} - (+)r^2\cos(2\beta+2\phi) - (+)r^2\cos 2\phi\} \\
&\quad + \tfrac{1}{4}(1+\cos^2\theta) - (+)2r\cos\theta\cos\beta.
\end{aligned} \tag{7}$$

Hence we obtain the variances

$$(\Delta\tilde{S}_{x,y})^2 = \langle\tilde{S}_{x,y}^2\rangle - \langle\tilde{S}_{x,y}\rangle^2. \tag{8}$$

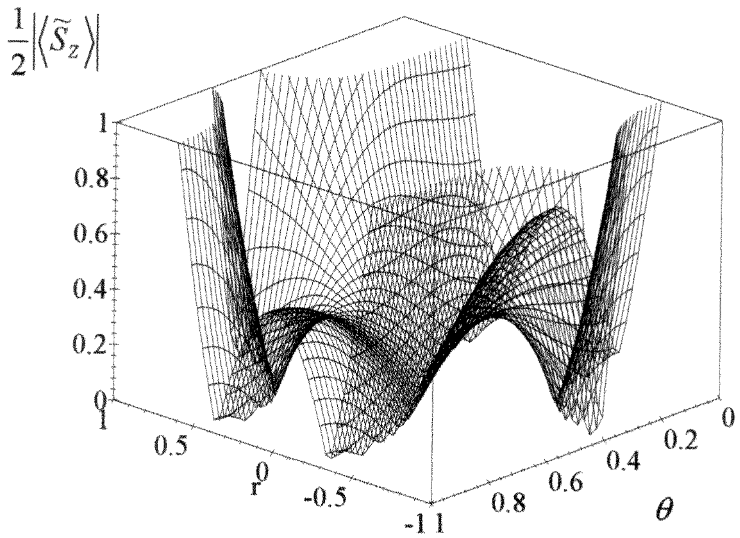


**Figure 1.** The plot of $\frac{1}{2}|\langle\tilde{S}_z\rangle|$ versus $r$ and $\theta$ for $2\phi + \beta = \pi$ and arbitrary $\beta$.

It should be noted that when the squeezing effect is removed, i.e. $\alpha = 0$, the uncertainties become

$$\Delta S_x^2 = \tfrac{1}{2}(1 - \sin^2\theta\cos^2\phi)$$
$$\Delta S_y^2 = \tfrac{1}{2}(1 - \sin^2\theta\sin^2\phi).$$

For the CSS, the equality in (1) holds either for $\theta = 0$ and arbitrary $\phi$ or arbitrary $\theta$ and $\phi = n\frac{\pi}{2}$ ($n$ is an integer). In addition, a local minimum of both sides of (1) is satisfied for $\theta = \frac{\pi}{2}$ and $\phi = n\frac{\pi}{2}$, and these four cases are considered the only MUS for the CSS [3]. However, these do not qualify as MUS according to the definition of Trifonov.

In the case of squeezed variances, if we minimize $|\langle\tilde{S}_z\rangle|$ with respect to the parameters $\theta, \phi, r$ and $\beta$ then we obtain $\beta + 2\phi = n\pi$; $r = \mp(\frac{1}{12}\tan^6\frac{\theta}{2} - \frac{1}{4}\tan^2\frac{\theta}{2} - \frac{1}{6})$; and four real values for $\theta$: $\tan\frac{\theta}{2} = \pm 1.5155$ and $\pm 0.659\,86$—where $+(-)$ in $r$ refers to even(odd) $n$. If we consider odd $n$, then we obtain two minima located at the points $\theta = 0.37\pi$, $r = -0.27$ and $\theta = 0.63\pi$, $r = 0.27$, as can be easily seen in figure 1. It is difficult to obtain the extreme points of the uncertainty products, i.e. the LHS of (1). We therefore plot it as functions of $r$ and $\theta$, where only one minimal value ($\theta = 0.63\pi$, $r = 0.27$) occurs at the same point as the RHS of (1), as seen in figure 2—where $\beta$ is taken to be $0.35\pi$, for which the equality in (1) is satisfied. With these values, the variances become $(\Delta\tilde{S}_x)^2 = 0.67$, $(\Delta\tilde{S}_y)^2 = 0.13$ and $\frac{1}{2}|\langle\tilde{S}_z\rangle| = 0.3$. Thus we see that both definitions for squeezing are satisfied.

It should be noted that the local minimum of the uncertainties product changes very slowly as the value of $\beta$ is changed. For even $n$ we obtain the same values for the local minima.

In this paper the SSS are defined through canonical transformations, which are constructed by analogy with the squeezed states of boson systems. For bosonic squeezing such a transformation results from parametric interaction with a strong classical pump. The simplest nonlinear interaction Hamiltonian with quadratic terms of the annihilation ($a$) and creation ($a^+$) boson operators and with interaction parameter $\chi$

$$H_I = \hbar(\chi a^2 + \chi^* a^{+^2}) \tag{9}$$

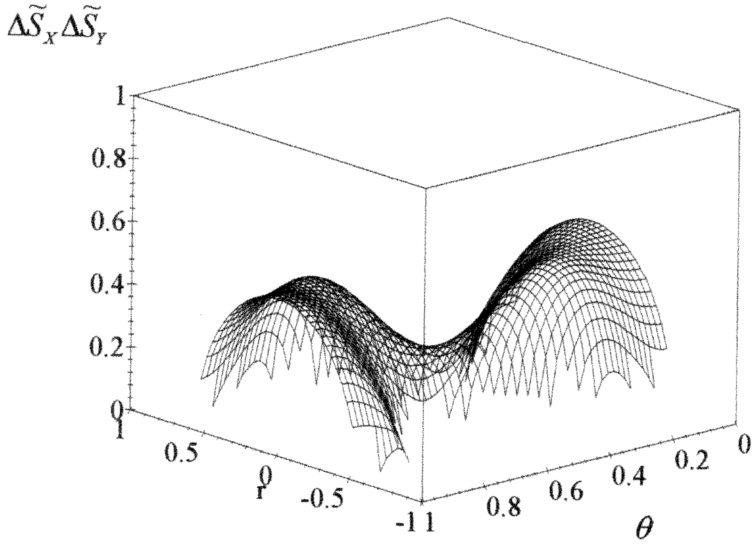


**Figure 2.** The plot of the uncertainties product $\Delta\tilde{S}_x\Delta\tilde{S}_y$ versus $r$ and $\theta$ for $2\phi+\beta=\pi\beta=0.35\pi$.

gives a unitary transformation $U = \exp(\mathrm{i}tH_I/\hbar)$, from which the squeezed states of light evolve either from the coherent states or equivalently the vacuum state. It also determines the general form of the well known unitary transformation that diagonalizes any Hamiltonian containing quadratic terms [12].

In the case of SSS, equation (4) is not arbitrary: the choice is similar to the multiatom squeezed states as defined by Barnett *et al* [13], containing correlated pairs of two-level atoms. Further to this implementation, one can consider the realization introduced by Kitagawa *et al* [4], where the concept of spin squeezing is established in terms of one- and two-axis twisting mechanisms. In particular, the two-axis twisting Hamiltonian gives the same canonical transformation as equation (4) with real $\alpha$, i.e.

$$H = \frac{\hbar\chi}{2\mathrm{i}}({S_+}^2 - S_-^2). \tag{10}$$

Therefore, we can implement our canonical transformation in terms of two-state systems, as discussed in detail by Kitagawa and Ueda [4]. These might be interferometers employing active elements such as four-wave mixers in their construction [14], where the interaction Hamiltonian equation (10) corresponds to coherent transfer of two particles at the same time.

## 3. Heisenberg interaction

We now consider two spin $\frac{1}{2}$ particles interacting via the anisotropic Heisenberg Hamiltonian

$$H_{12} = -[J_\parallel S_{1z}S_{2z} + J_\perp(S_{1x}S_{2x} + S_{1y}S_{2y})] \tag{11}$$

where $J_\parallel$ and $J_\perp$are exchange interactions, along the direction of the mean spin and perpendicular to it, respectively.

It is convenient to express equation (11) in terms of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$, which results in

$$H = -\tfrac{1}{2}[(S_z^2 - \tfrac{1}{2})(J_\parallel - J_\perp) + J_\perp(S^2 - \tfrac{3}{2})]. \tag{12}$$

In a recent paper [11], the exact ground-state spin configurations for two- and three-dimensional lattices were calculated by using this Hamiltonian with similar coherent spin states: where $\theta$ and $\phi$ are considered arbitrary parameters to construct all possible structures. In our calculation we consider $\theta$ and $\phi$ as variational parameters to obtain the optimal ground state.

We now transform the Hamiltonian (12) by the unitary operator $U$ to obtain squeezed spin effects such as

$$\tilde{H} = U^{-1}HU \tag{13}$$

using equation (4) and the terms up to the third power of $\alpha$ and $\alpha^*$. If we take the expectation value of this transformed Hamiltonian between the CSS $|\theta, \phi\rangle$, we then obtain a functional that depends on the parameters $\alpha$, $\alpha^*$, $\theta$ and $\phi$

$$\begin{aligned}\langle\theta, \phi|\tilde{H}|\theta, \phi\rangle = {} & \frac{1}{4}(J_\parallel - J_\perp)\sin^2\theta - J_\parallel\frac{1}{4} - 2|\alpha|^2\sin^4\frac{\theta}{2}(J_\parallel - J_\perp) \\ & + \tfrac{2}{3}\sin^2\theta(J_\parallel - J_\perp)|\alpha|^2(\alpha e^{2i\phi} + \alpha^* e^{-2i\phi}).\end{aligned} \tag{14}$$

By minimizing the last expression with respect to these parameters, we obtain four equations to solve. But upon choosing $\alpha = re^{i\beta}$ for convenience, as before, the $\phi$ derivation gives $\sin(2\phi + \beta) = 0$ and hence $2\phi + \beta = n\pi$. The other derivations result in

$$\frac{16}{3}\alpha e^{2i\phi} + \frac{8}{3}\alpha^* e^{-2i\phi} - 2\tan^2\frac{\theta}{2} = 0 \tag{15}$$

$$1 - 4|\alpha|^2\frac{\sin^2\frac{\theta}{2}}{\cos\theta} + \frac{8}{3}|\alpha|^2(\alpha e^{2i\phi} + \alpha^* e^{-2i\phi}) = 0. \tag{16}$$

The solutions to these equations are $\tan^2\frac{\theta}{2} = 0.853$ and $r = -0.25\tan^2\frac{\theta}{2} = -0.21$ for all $n$. With these values the interaction energy becomes

$$\langle\tilde{H}\rangle = -1.6425.10^{-3}J_\parallel - 0.2662J_\perp. \tag{17}$$

It should be noted that if the isotropic Heisenberg Hamiltonian is considered, i.e. $J_\parallel = J_\perp = J$, then one obtains $H = -\frac{1}{2}J(S^2 - \frac{3}{2})$, which is invariant under any unitary transformation. This does not mean that it is not possible to squeeze spins for the isotropic Heisenberg Hamiltonian: a modified unitary operator can be introduced in place of equation (4) and then it is more convenient to define the generalized CS as a direct product of the CSS taken in each lattice site, as will be discussed in section 4. Therefore we assume that the Hamiltonian in equation (12) shows an easy-axis anisotropy, that is $J_\parallel > J_\perp$. Then it is easily diagonalized and gives $-\frac{1}{4}J_\parallel$ and $(\frac{1}{4}J_\parallel - \frac{1}{2}J_\perp)$: here the first term is the ground-state energy and doubly degenerate. This can also be seen from equation (14) by taking $\theta$ and $\alpha$ to be zero. On the other hand, if the expectation value of H between the CS is minimized with respect to $\theta$, one obtains $\theta = \frac{\pi}{2}$, for which the CS energy becomes $-\frac{1}{4}J_\perp$—and it is exact. This is not surprising, since the spin $S$ pointing along the $z$-axis rotates through an angle $\theta$ about the axis $\boldsymbol{i}\sin\phi - \boldsymbol{j}\cos\phi$ [15]. This is further twisted by the squeezing transformation, which can be clearly seen in the quasiprobability distributions on the sphere for $S$ [4]. When compared with the result of the CSS, the energy appears to be reduced in the squeezed states. This justifies the general result that the squeezing effects cause more stable ground states. Contrary to the result of the CSS, equation (17) contains a small contribution from the component $J_\parallel$ of the Heisenberg interaction. This is a result of the angle $\theta$ now having a minimum value of $0.47\pi$ after the squeezing transformation. The result in equation (17) is obviously larger than the exact value, but how close it is depends on the magnitudes of $J_\parallel$ and $J_\perp$.

## 4. Conclusions

In this paper we have constructed SSS through canonical transformation analogous to the squeezed states for boson systems. Although this approach is no easier to use than others [4, 10], it has the advantage of being extended easily to two-mode squeezing as in the case of boson systems. It should be pointed out that one needs at least two spins to construct the SSS, which in fact includes the quantum mechanical correlation among these spins, while it is possible to produce the CSS from a single spin as clearly explained in [4]. Also in the same reference, the optimally squeezed states are obtained by changing the total $S$-spin, which gives the minimum variance for $S = 20$. In our case, we have considered the minimum number of elementary spins (two), and chosen coherent and squeezed parameters $\theta, \phi, \beta$ and $r$ as variables, so that the optimal values of variances are searched for the place where one variance is reduced and equation (1) is satisfied at the same time.

In section 3 we applied this method to the spin system interacting via the anisotropic Heisenberg Hamiltonian. Here we have considered $\theta, \phi, \beta$ and $r$ as trial parameters and calculated the ground-state energy by a variational method. The result shows that the squeezing effect indeed gives more a stable state than the CSS. The expansion in equation (5) is in powers of $\alpha$ and the $n$th term is proportional to $\frac{\alpha^n}{n!}$. Since $\alpha$ is generally smaller than unity (exactly $\alpha \cong 0.2$), it seems sufficient to consider a limited number of terms in the calculation. It should be noted that canonicality is obviously lost when the series in equation (5) are truncated, which is a drawback of this approach.

The approach of this work has been chosen to connect with the existing literature. It can be easily modified for a problem involving spin interaction. For example, for an interacting spin system in a lattice the squeezing transformation can be taken as

$$U = \exp \sum_{j,\delta} [\alpha_j S_j^+ S_{j+\delta}^+ - \alpha_j^* S_j^- S_{j+\delta}^-] \tag{18}$$

where $S_j^+$ and $S_j^-$ are the raising and lowering operators for each site, respectively and $\delta = \pm 1$ for nearest-neighbour interactions. This form is analogous to the two-mode squeezing of boson systems, which includes the correlation between the modes. In that case, it is also convenient to choose the CS in the following product form

$$\bigotimes_j \left( \cos \frac{\theta_j}{2} | \uparrow\rangle_j + \mathrm{e}^{\mathrm{i}\phi_j} \sin \frac{\theta_j}{2} | \downarrow\rangle_j \right). \tag{19}$$

Although this definition is more general than equation (2), it requires the handling of more variational parameters, which can be achieved on a finite system.

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## References

[1] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
[2] Altanhan T and Kandemir B S 1993 *J. Phys.: Condens. Matter* **5** 6729
Altanhan T and Kandemir B S 1997 *J. Phys.: Condens. Matter* **9** 427
Lo C F and Wong W H 1995 *Phys. Rev.* B **52** 3333
Kristoffersen A, Lo C F and Sollie R 1996 *Nuovo Cimento* **18** 931

[3] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458
[4] Kitagawa M and Ueda M 1993 *Phys. Rev.* A **47** 5138
[5] Klauder J and Skagerstam B S 1985 *Coherent States* (Singapore: World Scientific)
[6] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[7] Radcliffe J M 1971 *J. Phys. A: Math. Gen.* **4** 27 0
[8] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
Perelomov A M 1986 *Coherent States and their Applications* (Berlin: Springer)
[9] Aragone C, Guerri G, Salamo S and Tani J L 1974 *J. Phys. A: Math. Gen.* **15** L149
Aragone C, Chalbaud E and Salamo S 1976 *J. Math. Phys.* **17** 1963
[10] Trifonov D A 1993 *J. Math. Phys.* **34** 100
Trifonov D A 1994 *J. Math. Phys.* **35** 2297
Trifonov D A 1994 *Phys. Lett.* A **187** 284
[11] Belorizky E 1995 *Solid State Commun.* **96** 853
[12] Kitagawa M and Yamamoto Y 1986 *Phys. Rev.* A **34** 3974
Kitagawa M, Imoto N and Yamamoto Y 1987 *Phys. Rev.* A **35** 52 070
Kitagawa M and Ueda M 1991 *Phys. Rev. Lett.* **67** 1852
[13] Barnett S M and Dupertuis M A 1987 *J. Opt. Soc. Am.* B **4** 505
Dupertuis A, Barnett S M and Stenholm S 1987 *J. Opt. Soc. Am.* B **4** 1124
[14] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev.* A **33** 4033
[15] Barnett S M and Radmore P M 1997 *Methods in Theoretical Quantum Optics* (Oxford: Oxford University Press)